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by

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Periodic Solutions of a Class of Hyperbolic Equations Containing a Small Parameter

Jack K. Hale

1. Introduction. The purpose of the present paper is to discuss a method for the determination of solutions $u(t,x)$ and $v(y,z)$ of the problems

$$(1.1)\qquad \begin{aligned} u_{tt}-u_{xx} &= \epsilon\, g(t,x,u,u_t,u_x), \\ u(t,0) &= u(t,\pi) = 0 , \end{aligned}$$

and

$$(1.2)\qquad v_{yz} = \epsilon\, f(y,z,v,v_y,v_z),$$

respectively, with $u(t+2\pi,x) = u(t,x)$, $v(y+2\pi,z) = v(y,z) = v(y,z+2\pi)$ provided that g is 2π-periodic in t and f is 2π-periodic in y,z and ϵ is a small real parameter. Equation (1.1) has been discussed by many authors and the reader may consult the paper of Vejvoda [6] for a survey of results as well as an extensive bibliography. Equation (1.2) has been discussed extensively by L. Cesari [2].

Our aim in this paper is to show that the method used by L. Cesari and the author for similar problems in ordinary differential equations can be extended in a completely analogous way to equations (1.1) and (1.2). In fact, after the preliminary discussion of the

Fredholm alternative for (1.1), (1.2) given, respectively, in sections 2.1 and 3.1, the reader will observe that the results as well as the techniques follow very closely Chapter 6 of the monograph [3]. We obtain a set of bifurcation or determining equations for equations (1.1), (1.2) which are equations for an unknown function satisfying the homogeneous equation and represent necessary and sufficient conditions for the existence of periodic solutions of the type stated previously. One can then apply the implicit function theorem to obtain sufficient conditions for the existence of periodic solutions (see Theorems 4 and 9). These sufficient conditions coincide with the ones obtained by O. Vejvoda [7, Theorem 1.4.1] by an application of a procedure more similar to the usual method of Poincaré in ordinary differential equations.

L. Cesari [2] also obtained bifurcation equations for (1.2) and the method used in this paper is very similar to the one used by L. Cesari. The unknown function in the bifurcation equations of Cesari represent the initial values of the solution whereas our unknown function is the projection of the solution onto a solution of the homogeneous equation. This latter type of unknown injects more geometry into the problem and is completely analogous to the method of Cesari and the author for ordinary differential equations. The reader will find it instructive to compare this method with the recent work of H. Antosiewicz [1].

In the following, we shall let R^n be a normed n-dimensional real vector space and let C_k be the space of all functions mapping R^2 into R^1 which are bounded and continuous together with all derivatives up through order k. For any φ in C_k, the norm, $\|\varphi\|_k$, is defined by

$$\|\varphi\|_k = \sum_{j=0}^{k} \sum_{\nu+\mu=j} \sup_{R^2} \left| \frac{\partial^{\mu+\nu}\varphi(x,y)}{\partial x^{\mu}\partial x^{\nu}} \right| .$$

We shall also use the same notation C_k when the functions map R^1 into R^1. It will always be clear from the context whether the domain is R^2 or R^1.

2. The wave equation.

2.1. The linear equation. Consider the classes of functions C_k^*, T and M defined by

$$C_k^* = \{\varphi : \varphi_x^{(j)}(t,x) \stackrel{\text{def}}{=} \partial^j\varphi(t,x)/\partial x^j \text{ is in } C_o \text{ for } j=0,1,2,\ldots,k\}$$

$$T = \{\varphi : \varphi(t+2\pi,x) = \varphi(t,x) = \varphi(t,x+2\pi), \varphi(t,-x) = -\varphi(t,x)\}$$

$$M = \{\varphi : \varphi(t,x) = p(x+t)-p(-x+t), p(t+2\pi) = p(t)\} .$$

For any φ in C_k^*, we define $\|\varphi\|_k^* = \sup \{\|\varphi_x^{(j)}\|_o, j=o,1,\ldots,k\}$. For any φ in $T \cap C_k^*$, $k \geq 0$, we define the element $Q\varphi$ in $M \cap C_k$ by

$$(2.2) \qquad (Q\varphi)(t,x) = \frac{1}{2\pi}\int_o^{2\pi} \varphi(s,x+t-s)ds - \frac{1}{2\pi}\int_o^{2\pi} \varphi(s,-x+t-s)ds ,$$

$$0 \leq t,\ x \leq 2\pi .$$

It follows easily from the definition that Q is a projection operator mapping $T \cap C_k^*$ onto $M \cap C_k$. Finally, we designate by $M^\perp$ the following set:

$$(2.3) \qquad M^\perp = \{\varphi \text{ in } T \cap C_o : Q\varphi = 0\} .$$

Notice that any element φ in $T \cap C_k^*$ can be represented as $\varphi = Q\varphi + (I-Q)\varphi$ and that φ in $M^\perp \cap C_k^*$ implies $\varphi = (I-Q)\varphi$

and, therefore, $\int_0^{2\pi} \varphi(s,-s)ds = 0$. The latter relation follows because φ in $M^{\perp} \cap C_k^*$ implies

$$\int_0^{2\pi} \varphi(t,-t)dt = \int_0^{2\pi} [(I-Q)\varphi](t,-t)dt = \frac{1}{2\pi}\int_0^{2\pi} [\int_0^{2\pi} \varphi(s,2t-s)dt]ds$$

$$= \frac{1}{2\pi}\int_0^{2\pi} [\int_0^{2\pi} \varphi(s,u)du]ds = 0$$

since $\varphi(s,-u) = -\varphi(s,u)$.

<u>Lemma 1</u>. For any nonnegative integer k and any φ in $T \cap C_k^*$, the following statements are equivalent:

i) $\varphi \in M^{\perp} \cap C_k^*$;

ii) $\int_0^{2\pi} \varphi(s,y-s)ds = 0,\ 0 \leqq y \leqq 2\pi$;

iii) $\int_0^{2\pi}\int_0^{\pi} \varphi(t,x)\gamma(t,x)dxdt = 0$ for all γ in $M \cap C_k$.

<u>Proof</u>: If $\varphi \in M^{\perp} \cap C_k^*$, then $(Q\varphi)(t,x) = 0$ for all t,x, and $\int_0^{2\pi}\varphi(s,-s)ds = 0$. In particular, $2\pi(Q\varphi)(t,-t) = -\int_0^{2\pi}\varphi(s,2t-s)ds = 0$ for all t. Therefore, i) implies ii). If ii) is satisfied, then i) is obviously satisfied.

To prove that ii) is equivalent to iii), suppose that

$\gamma(t,x) = p(x+t)-p(-x+t)$. Then

$$\int_0^{2\pi}\int_0^{\pi} \varphi(t,x)[p(x+t)-p(-x+t)]dxdt$$

$$= \int_0^{2\pi}\int_t^{t+\pi} \varphi(t,y-t)p(y)dydt - \int_0^{2\pi}\int_{t-\pi}^{t} \varphi(t,t-y)p(y)dydt$$

$$= \int_0^{2\pi}\int_t^{t+\pi} \varphi(t,y-t)p(y)dydt + \int_0^{2\pi}\int_{t-\pi}^{t} \varphi(t,y-t)p(y)dydt$$

$$= \int_0^{2\pi}\int_{t-\pi}^{t+\pi} \varphi(t,y-t)p(y)dydt$$

$$= \int_{-\pi}^{\pi} \left[\int_0^{2\pi} \varphi(t,y-t)dt \right] p(y)dy \; .$$

and this relation implies ii) is equivalent to iii).

The following property is well known.

<u>Lemma 2</u>. The set $M \cap C_2$ coincides with the set of solutions in C_2 of the problem

$$u_{tt}-u_{xx} = 0 \tag{2.4}$$
$$u(t,0) = u(t,\pi) = 0 \; , \; 0 < x < \pi, \; t > 0,$$

Now consider a continuous function $\varphi(t,x), \varphi(t+2\pi,x) = \varphi(t,x)$, $0 \leq x \leq \pi$, $\varphi(t,0) = \varphi(t,\pi) = 0$ and the associated boundary value problem

$$
(2.5) \qquad \begin{aligned} & u_{tt}-u_{xx} = \varphi(t,x) \\ & u(t,0) = u(t,\pi) = 0 \\ & u(t+2\pi,x) = u(t,x) \ , \ 0 < x < \pi \ , \ t > 0 \ . \end{aligned}
$$

By extending the function $\varphi(t,x)$ as an odd 2π-periodic function of x , and keeping the same notation for the extension of φ, the above problem is equivalent to the following:

$$
(2.6) \qquad \begin{aligned} & u_{tt}-u_{xx} = \varphi(t,x) \\ & u(t+2\pi,x) = u(t,x) = u(t,x+2\pi) \\ & u(t,x) = -u(t,-x) \ , \ -\infty < x < \infty \ , \ t > 0 \ . \end{aligned}
$$

Lemma 3. For any given integer $k \geqq 1$ and a given φ in $T \cap C_k^*$, the problem (2.6) has a solution if and only if $\varphi \in M^{\perp} \cap C_k^*$. Furthermore, if $\varphi \in M^{\perp} \cap C_k^*$, then there exists a unique solution of (2.6) in $M^{\perp} \cap C_{k+1}$. If this unique solution is designated by $\mathscr{K}(t,x)\varphi$, $0 \leqq t$, $x \leqq 2\pi$, then $\mathscr{K}(\cdot,\cdot)$ is a linear operator mapping $M^{\perp} \cap C_k^*$ into $M^{\perp} \cap C_{k+1}$ and there is a constant K such that

$$
(2.7) \qquad \|\mathscr{K}(\cdot,\cdot)\varphi\|_{k+1} \leqq K\|\varphi\|_k^* \ .
$$

Proof: The first part of the lemma follows from a result of Vejvoda [6, p.365] and Lemma 2. For the sake of completeness, we include a proof of this fact here. For any φ in $T \cap C_k^*$, $k \geqq 1$, consider the function

$$U(t,x) = \frac{1}{2}\int_0^t \int_{x-t+\theta}^{x+t-\theta} \varphi(\theta,\xi)d\xi d\theta$$

which belongs to C_{k+1}, satisfies $U_{tt}-U_{xx} = \varphi(t,x)$ and is clearly periodic in x. Also, $\varphi(t,-x) = -\varphi(t,x)$ implies $U(t,-x) = -U(t,x)$. To prove the first part of the lemma, it is sufficient to show that $U(t,x) = U(t+2\pi,x)$ if and only if φ belongs to $M^{\perp} \cap C_k^*$. A straightforward computation making use of $\varphi(t,-x) = -\varphi(t,x)$ yields $U(t+2\pi,x) = U(t,x)$ for all t,x if and only if

$$\psi(t,x) \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_0^{2\pi} \int_{x-t+\theta}^{x+t-\theta} \varphi(\theta,\xi)d\xi d\theta = 0$$

for all t,x. But since $\psi(t,x) \in C_1$ we will obtain $\psi(t,x) = 0$ for all t,x if and only if $\psi_t(t,x) = 0$, $\psi_x(t,x) = 0$ for all t,x and there is a value of t,x for which $\psi(t,x) = 0$. But, using the fact that $\varphi(t,-x) = -\varphi(t,x)$, we obtain

$$\psi_t(t,x) = (Q\varphi)(t,x)$$
$$\psi_x(t,x) = (Q\varphi)(t,x) + \frac{1}{\pi}\int_0^{2\pi} \varphi(\theta,-x+t-\theta)d\theta$$

If $\psi_t(t,x) = 0 = \psi_x(t,x)$ for all t,x, then $\psi(t,x) = $ constant and

$$\psi(t,t) = \frac{1}{2\pi}\int_{0}^{2\pi}\int_{\theta}^{2t-\theta} \varphi(\theta,\xi)d\xi d\theta$$

$$= -\frac{1}{2\pi}\int_{0}^{2\pi}\int_{-\theta}^{-2t+\theta} \varphi(\theta,-\xi)d\xi d\theta$$

$$= -\frac{1}{2\pi}\int_{0}^{2\pi}\int_{-2t+\theta}^{-\theta} \varphi(\theta,\xi)d\xi d\theta = -\psi(t,-t)$$

which implies $\psi(t,x) = 0$ for all t,x. But $\psi_t(t,x) = 0 = \psi_x(t,x)$ for all t,x if and only if φ is in $M^{\perp} \cap C_k^*$ and, therefore, $U(t+2\pi,x) = U(t,x)$ if and only if φ is in $M^{\perp} \cap C_k$. This completes the proof of the first part of the lemma.

Obviously, if φ belongs to $M^{\perp} \cap C_k^*$ then $\mathscr{K}(\cdot,\cdot)\varphi \overset{\text{def}}{=} U-QU$ is a solution of (2.5) in $M^{\perp} \cap C_{k+1}$ and is the only solution with this property. The estimate (2.7) is obtained by simple differentiations and integrations. This completes the proof of Lemma 3.

2.2. <u>The nonlinear wave equation.</u> Consider the problem

$$(2.8)\qquad \begin{aligned} &u_{tt}-u_{xx} = \epsilon g(t,x,u,u_t,u_x) \\ &u(t+2\pi,x) = u(t,x) = u(t,x+2\pi) \\ &u(t,x) = -u(t,-x) \end{aligned}$$

where ϵ is a real parameter and

$$g(t+2\pi,x,u,p,q) = g(t,x,u,p,q) = g(t,x+2\pi,u,p,q) \tag{2.9}$$
$$g(t,x,u,p,q) = -g(t,-x,-u,-p,q)$$

for all t,x,u,p,q in the region $\Omega(R)$ defined by

$$\Omega(R) = \{(t,x,u,p,q): -\infty < t,x < \infty, |u|+|p|+|q| \leq R\}. \tag{2.10}$$

Obtaining a solution of (2.8) is equivalent to obtaining a solution of the problem

$$u_{tt}-u_{xx} = \epsilon\, g(t,x,u,u_t,u_x)$$
$$u(t,0) = u(t,2\pi), \quad u(t+2\pi,x) = u(t,x)$$

provided that g is periodic in t of period 2π and

$$g(t,0,0,0,q) = g(t,\pi,0,0,q) = 0 .$$

Therefore, we restrict our discussion to (2.8).

In the following, we shall sometimes impose one of the following hypotheses:

(A_1) $g(t,x,u,p,q)$ is continuous in $\Omega(R)$ together with Lipschitz continuous first derivatives with respect to x,u,p,q in $\Omega(R)$;

(A_2) $g(t,x,u,p,q)$ is continuous in $\Omega(R)$, has continuous first and Lipschitz continuous second derivaties with respect to x,u,p,q in $\Omega(R)$;

(A_3) g depends only upon t,x,u, is continuous in $\Omega(R)$ together with its first, second and third derivatives with respect to x,u in $\Omega(R)$.

For a fixed γ in $M \cap C_2$ and for given positive constants a,b, $a < b < R$, let

$$(2.11) \qquad T_2(\gamma,a,b) = \{\psi \text{ in } T \cap C_2 \colon Q\psi = \gamma,\ \|\gamma\|_2 \leq a,\ \|\psi\|_2 \leq b\}.$$

If φ is any function in $T_2(\gamma,a,b)$, then we will let $G(\cdot,\cdot,\varphi)$ designate the function

$$(2.12) \qquad G(t,x,\varphi) \overset{\text{def}}{=} g(t,x,\varphi(t,x),\varphi_t(t,x),\varphi_x(t,x)),\ 0 \leq t,x \leq 2\pi.$$

<u>Theorem 1.</u> If g satisfies (A_1) and $a < b < R$ are given positive constants, then there is an $\epsilon_1 > 0$ with the following property: corresponding to each γ in $M \cap C_2$, $\|\gamma\|_2 \leq a$ and to each ϵ, $|\epsilon| \leq \epsilon_1$, there is a unique function $\Gamma = \Gamma(\gamma,\epsilon)$ in $T_2(\gamma,a,b)$, continuous in γ and ϵ, $\Gamma(\gamma,0) = \gamma$, such that Γ satisfies the relation

$$\text{(2.13)}\quad \Gamma_{tt}-\Gamma_{xx} = \epsilon\, G(t,x,\Gamma) - \epsilon\, Q\, G(\cdot,\cdot,\Gamma)(t,x)$$

where G is defined in (2.12). The function $\Gamma(\gamma,\epsilon)$ can be obtained by the method of successive approximations

$$\text{(2.14)}\quad \begin{aligned} u^{(0)} &= \gamma \\ u^{(n+1)}(t,x) &= \gamma(t,x) + \epsilon\,\mathscr{K}(t,x)(I-Q)G(\cdot,\cdot,u^{(n)}), \\ & \qquad n = 0,1,2,\ldots \end{aligned}$$

where $\mathscr{K}(\cdot,\cdot)$ is defined in Lemma 3. Finally, $\Gamma(\gamma,\epsilon)$ is Lipschitzian with respect to γ uniformly with respect to ϵ for $\|\gamma\|_2 \leq a$, $|\epsilon| \leq \epsilon_1$.

Proof: If g satisfies (A_1), then it is clear that G defined in (2.12) belongs to C_1^* for any φ in $T_2(\gamma,a,b)$. If $\mathscr{K}(\cdot,\cdot)$ is the operator defined in Lemma 3, we define an operator $\mathscr{F}$ by

$$(\mathscr{F}\varphi)(t,x) = \gamma(t,x) + \epsilon\,\mathscr{K}(t,x)(I-Q)G(\cdot,\cdot,\varphi)\ .$$

The hypotheses on g imply that there are constants K_1, K_2 such that

$$\|G(\cdot,\cdot,\varphi)\|_1^* \leq K_1,\ \|G(\cdot,\cdot,\varphi)-G(\cdot,\cdot,\psi)\|_1^* \leq K_2\|\varphi-\psi\|_2\ ,$$

for any φ,ψ in $T_2(\gamma,a,b)$. From Lemmas 1 and 2, the fixed points of $\mathcal{F}$ in $T_2(\gamma, a, b)$ coincide with the solution of (2.13) in $T_2(\gamma, a, b)$. Therefore, it suffices to show that there is an $\epsilon_1 > 0$ such that the operator $\mathcal{F}$ has a unique fixed point in $T_2(\gamma, a, b)$ for $|\epsilon| \leq \epsilon_1$. We prove this by showing that $\mathcal{F}$ is a contraction mapping of $T_2(\gamma, a, b)$ into itself.

For the above constants K_1, K_2 and the constant K in Lemma 3, choose $\epsilon_1 > 0$ such that

$$a + 2K_1 K\epsilon_1 < b \ , \ 2K_2 K\epsilon_1 < 1 \ .$$

For $|\epsilon| \leq \epsilon_1$, we obtain from the definition of $\mathcal{F}$ and Lemma 3 that

$$\begin{aligned} \|\mathcal{F}\varphi\|_2 &\leq \|\gamma\|_2 + \epsilon_1 \|\mathcal{K}(\cdot,\cdot)(I-Q)G(\cdot,\cdot,\varphi)\|_2 \\ &\leq a + 2K\epsilon_1 \|G(\cdot,\cdot,\varphi)\|_1^* \\ &\leq a + 2K_1 K\epsilon_1 < b \ . \end{aligned}$$

Also,

$$\begin{aligned} \|\mathcal{F}\varphi - \mathcal{F}\psi\|_2 &\leq \epsilon_1 \|\mathcal{K}(\cdot,\cdot)(I-Q)[G(\cdot,\cdot,\varphi) - G(\cdot,\cdot,\psi)]\|_2 \\ &\leq 2K\epsilon_1 \|G(\cdot,\cdot,\varphi) - G(\cdot,\cdot,\psi)\|_1^* \\ &\leq 2K_2 K\epsilon_1 \|\varphi - \psi\|_2 \end{aligned}$$

for all φ, ψ in $T_2(\gamma, a, b)$. This shows that $\mathcal{F}$ is a contraction mapping of $T_2(\gamma, a, b)$ into itself. Therefore, $\mathcal{F}$ has a unique fixed point $\Gamma(\gamma, \epsilon)$ in $T_2(\gamma, a, b)$ and it is obtainable by the method of successive approximations (2.14). The function $\Gamma(\gamma, \epsilon)$ is clearly continuous in ϵ for $|\epsilon| \leqq \epsilon_1$ and $\Gamma(\gamma, 0) = \gamma$. Furthermore, for any γ, δ in $M \cap C_2$ with $\|\gamma\|_2, \|\delta\|_2 \leqq a$,

$$\|\Gamma(\gamma,\epsilon)-\Gamma(\delta,\epsilon)\|_2 = \|\mathcal{F}\Gamma(\gamma,\epsilon)-\mathcal{F}\Gamma(\delta,\epsilon)\|_2$$

$$\leqq \|\gamma-\delta\|_2 + 2K_2K\,\epsilon_1\|\Gamma(\gamma,\epsilon)-\Gamma(\delta,\epsilon)\|$$

for $|\epsilon| \leqq \epsilon_1$. Since $2K_2K\,\epsilon_1 < 1$, this implies $\Gamma(\gamma, \epsilon)$ is Lipschitzian with respect to γ uniformly with respect to ϵ. Consequently, $\Gamma(\gamma, \epsilon)$ is jointly continuous in γ, ϵ and the theorem is proved.

<u>Theorem 2</u>. Suppose g satisfies (A_1) and $a < b < R, \epsilon_1, \Gamma(\gamma, \epsilon)$ are the quantities given in Theorem 1. If there exist an $\epsilon_2 \leqq \epsilon_1$ and a function $\gamma(\epsilon)$ in $M \cap C_2$, $|\gamma(\epsilon)| \leqq a$, $|\epsilon| \leqq \epsilon_2$, such that

$$QG(\cdot,\cdot,\Gamma(\gamma(\epsilon),\epsilon)) = 0 \,, \tag{2.15}$$

then $\Gamma(\gamma(\epsilon),\epsilon)$ is a solution of (2.8) for $|\epsilon| \leqq \epsilon_2$. Conversely, if (2.8) has a solution $u(t,x,\epsilon)$ which is continuous in t, x, ϵ together with all first and second derivatives with respect to t, x for $0 \leqq t$, $x \leqq 2\pi$, $|\epsilon| \leqq \epsilon_2$, $\|u(\cdot,\cdot,\epsilon)\|_2 < b$, $\|Qu(\cdot,\cdot,\epsilon)\|_2 \leqq a$,

$0 \leq |\epsilon| \leq \epsilon_2$, then $u(t,x,\epsilon) = \Gamma(t,x,\gamma(\epsilon),\epsilon)$ where Γ is the function given in Theorem 1, $Qu(\cdot,\cdot,\epsilon) = \gamma(\epsilon)$ and $\gamma(\epsilon)$ satisfies (2.15).

Proof: The first part of the theorem is obvious. To prove the second part, let $Qu(\cdot,\cdot,\epsilon) = \gamma(\epsilon)$. Since u is a solution of (2.8), it follows that

$$(I-Q)[u_{tt}-u_{xx}-\epsilon\, G(\cdot,\cdot,u)] = 0$$

$$Q[u_{tt}-u_{xx}-\epsilon\, G(\cdot,\cdot,u)] = 0\ .$$

One easily shows that $Q(u_{tt}) = (Qu)_{tt}$, $Q(u_{xx}) = (Qu)_{xx}$. Since $Qu = \gamma(\epsilon)$, $\gamma(\epsilon)$ in $M \cap C_2$, it follows that $Q(u_{tt}-u_{xx}) = 0$. Therefore, the above equations are equivalent to

$$u_{tt}-u_{xx} = \epsilon\, G(\cdot,\cdot,u) - \epsilon\, Q\, G(\cdot,\cdot,u)$$

$$QG(\cdot,\cdot,u) = 0$$

for $0 \leq |\epsilon| \leq \epsilon_2$. Theorem 1 implies that $u = \Gamma(\gamma(\epsilon),\epsilon)$ from the first equation and the second equation implies $\gamma(\epsilon)$ satisfies (2.15). This completes the proof of the theorem.

Theorem 3. Suppose g satisfies (A_2) and $a < b < R$, $\epsilon_1, \Gamma(\gamma,\epsilon)$ are the quantities given in Theorem 1. Then there exists an ϵ_3,

$0 < \epsilon_3 \leqq \epsilon_1$ such that $\Gamma(\gamma, \epsilon)$ is continuously differentiable with respect to γ for $\|\gamma\|_2 < a$, $|\epsilon| \leqq \epsilon_3$. Furthermore, the derivative of $\Gamma(\gamma, \epsilon)$ with respect to γ at $\epsilon = 0$ is the identity operator.

Proof: In the proof, we will not write the explicit dependence of Γ upon ϵ and will use the simpler notation $\Gamma(\gamma)$. For any γ in $M \cap C_2$, $\|\gamma\|_2 < a$, we need to show (see, for example, [4]) that there is a continuous linear operator U_γ mapping $M \cap C_2$ into $T \cap C_2$ such that for every $r > 0$, there is an $s > 0$ such that

$$\|\Gamma(\gamma + \Delta)-\Gamma(\gamma) - U_\gamma\Delta\|_2 \leqq r\|\Delta\|_2$$

for Δ in $M \cap C_2$ and $\|\Delta\|_2 < s$. If we choose $\|\Delta\|_2$ small enough, say $\|\Delta\|_2 < s_1$, then $\Gamma(\gamma + \Delta)$ will be well defined from Theorem 1. If we let $w^\Delta = \Gamma(\gamma + \Delta)-\Gamma(\gamma)$, use the definition of $G(\cdot,\cdot,\varphi)$ in (2.12) and the mean value theorem, we obtain

$$G(t,x,w^\Delta+\Gamma(\gamma))-G(t,x,\Gamma(\gamma))$$

$$= g_u^\Delta(t,x)w^\Delta(t,x)+g_p^\Delta(t,x)w_t^\Delta(t,x)+g_q^\Delta(t,x)w_x^\Delta(t,x)$$

where

$$g_u^{\Delta}(t,x) = g_u(t,x,\Gamma(t,x,\gamma+\Delta_1),\Gamma_t(t,x,\gamma+\Delta),\Gamma_x(t,x,\gamma+\Delta))$$

$$g_p^{\Delta}(t,x) = g_p(t,x,\Gamma(t,x,\gamma),\Gamma_t(t,x,\gamma+\Delta_2),\Gamma_x(t,x,\gamma+\Delta))$$

$$g_q^{\Delta}(t,x) = g_q(t,x,\Gamma(t,x,\gamma),\Gamma_t(t,x,\gamma),\Gamma_x(t,x,\gamma+\Delta_3))$$

and $\Delta_1, \Delta_2, \Delta_3$ approach zero as Δ approaches zero. Notice that g_u^o, g_p^o, g_q^o depend only upon γ and there is a constant K_4 such that $\|g_u^{\Delta}\|_1^*$, $\|g_p^{\Delta}\|_1^*$, $\|g_q^{\Delta}\|_1^* \leq K_4$ for $\|\Delta\|_2 < s_1$. Also, from the continuity of $\Gamma(\gamma)$ in γ, we have

$$\|g_u^{\Delta} - g_u^o\|_1^*,\ \|g_p^{\Delta} - g_p^o\|_1^*,\ \|g_q^{\Delta} - g_q^o\|_1^* \leq \nu(\|\Delta\|_2)$$

where $\nu(s) \to 0$ as $s \to 0$.

Consider now the integral equation

$$V(t,x) = \varphi(t,x) + \epsilon\,\mathscr{K}(t,x)(I-Q)[g_u^oV + g_p^oV_t + g_q^oV_x],$$

where $\mathscr{K}(\cdot,\cdot)$ is defined in Lemma 3 and φ is an arbitrary element in $M \cap C_2$. For $|\epsilon| \leq \epsilon_3 \leq \epsilon_1$, $2K_4K\epsilon_3 < 1$ and any φ in $M \cap C_2$, it is a simple matter to show (by an argument similar to that in the proof of Theorem 1) that this equation has a unique solution $V(\varphi)$ in $T \cap C_2$ which satisfies $\|V(\varphi)\|_2 \leq (1-2K_4K\epsilon_3)^{-1}\|\varphi\|_2$. Furthermore, by the uniqueness, one obtains $V(k\varphi) = kV(\varphi)$, $V(\varphi+\psi) = V(\varphi) + V(\psi)$ for all constants k and all φ, ψ in $M \cap C_2$. Therefore $V(\varphi)$ is a continuous linear mapping of $M \cap C_2$ into $T \cap C_2$ and we designate this mapping by U_γ. Notice that U_γ is equal to the identity for $\epsilon = 0$.

With the above definition of w^Δ, U_γ and $\|\Delta\|_2 < s_1$, $|\epsilon| \leq \epsilon_3$, we have

$$w^\Delta - U_\gamma \Delta = \epsilon \mathcal{K}(\cdot,\cdot)(I-Q)[g_u^0(w^\Delta - U_\gamma \Delta) + g_p^0(w_t^\Delta - U_\gamma \Delta_t) + g_q^0(w_x^\Delta - U_\gamma \Delta_x)]$$

$$+ \epsilon \mathcal{K}(\cdot,\cdot)(I-Q)[(g_u^\Delta - g_u^0)w^\Delta + (g_p^\Delta - g_p^0)w^\Delta + (g_q^\Delta - g_q^0)w_x^\Delta] .$$

Using all of the previous estimates and the fact (from Theorem 1) that $\Gamma(\gamma)$ is Lipschitzian in γ with some Lipschitz constant K independent of ϵ, we obtain

$$\|w^\Delta - U_\gamma \Delta\|_2 \leq 2K_4 K \epsilon_3 \|w^\Delta - U_\gamma \Delta\|_2 + 2K_5 K \epsilon_3 \nu(\|\Delta\|_2)\|\Delta\|_2$$

and, thus,

$$\|w^\Delta - U_\gamma \Delta\|_2 \leq (1 - 2K_4 K \epsilon_3)^{-1} 2K_5 K \epsilon_3 \nu(\|\Delta\|_2)\|\Delta\|_2 .$$

This completes the proof of the theorem.

Notice that nothing is changed in the above theory if g depends continuously upon ϵ. We will use this remark in the examples.

We will refer to equations (2.15) as the <u>bifurcation equations</u> or <u>determining equations</u> for problem (2.8) and a solution

$\gamma(\epsilon)$ of these equations which belongs to $M \cap C_2$ is a necessary and sufficient condition (in the sense described by Theorem 2) for the existence of a solution to (2.8) for ϵ sufficiently small.

A more convenient form of the bifurcation equations can be obtained by the following argument. A necessary and sufficient condition for the existence of a solution of (2.8) is that $QG(\cdot,\cdot,\Gamma(\gamma,\epsilon)) = 0$ where $\Gamma(\gamma,\epsilon)$ is defined in Theorem 1. On the other hand, Lemma 1 yields the result that this is equivalent to saying that γ satisfies

$$H(\gamma,\epsilon) = 0,$$

$$(2.16) \qquad 2\pi H(\gamma,\epsilon)(y) \overset{\text{def}}{=} \int_0^{2\pi} g(s,y-s,\Gamma(s,y-s,\gamma,\epsilon),\Gamma_t(s,y-s,\gamma,\epsilon),\Gamma_x(s,y-s,\gamma,\epsilon))ds,$$

$$0 \leqq y \leqq 2\pi .$$

For later reference, the explicit formula for $H(\gamma,0)$ is

$$(2.17) \qquad 2\pi H(\gamma,0)(y) = \int_0^{2\pi} g(s,y-s,\gamma(s,y-s),\gamma_t(s,y-s),\gamma_x(s,y-s))ds$$

$$0 \leqq y \leqq 2\pi$$

which, by the way, can be calculated without any successive approximations whatsoever. Also, if the conditions of Theorem 3 are satisfied, then it is a simple matter to show that the derivative $H(\gamma,0)$ of $H(\gamma,0)$ with respect to γ is given by

$$(2.18) \qquad 2\pi[H'(\gamma,0)\Delta](y) =$$

$$= \int_0^{2\pi} [g_u(s,y-s,\gamma(s,y-s),\gamma_t(s,y-s),\gamma_x(s,y-s))\Delta(s,y-s)$$

$$+ g_p(s,y-s,\gamma(s,y-s),\gamma_t(s,y-s),\gamma_x(s,y-s))\Delta_t(s,y-s)$$

$$+ g_q(s,y-s,\gamma(s,y-s),\gamma_t(s,y-s),\gamma_x(s,y-s))\Delta_x(s,y-s)]ds \ ,$$

$$0 \leqq y \leqq 2\pi \ .$$

If g satisfies (A_1), then $H(\gamma,\epsilon)$ is a continuous mapping of $(M \cap C_2) \times [-\epsilon_1,\epsilon_1]$ into the subspace of C_1 consisting of 2π-periodic functions. If g satisfies (A_2), then $H(\gamma,\epsilon)$, $H'(\gamma,\epsilon)$ are continuous mappings of $(M \cap C_2) \times [-\epsilon_1,\epsilon_1]$ into the subspace of C_1 consisting of 2π-periodic functions. If g satisfies (A_3), then $H(\gamma,\epsilon)$, $H'(\gamma,\epsilon)$ are continuous mappings of $(M \cap C_2) \times [-\epsilon_1,\epsilon_1]$ into the subspace of C_2 consisting of 2π-periodic functions.

By using these remarks and the implicit function theorem in Banach spaces [4], we immediately obtain the following result which was previously discovered by Vejvoda [7,Theorem 4.1.1] by an application of a procedure more similar to the usual method of Poincaré in ordinary differential equations.

<u>Theorem 4</u>. Suppose g satisfies (A_2) and $H(\gamma,0)$, $H'(\gamma,0)$ are defined by (2.17),(2.18). If there exists a γ_0 in $M \cap C_2$,

$\|\gamma_0\|_2 < a$, such that $H(\gamma_0,0) = 0$ and $H'(\gamma_0,0)$ has a continuous inverse which maps the subspace of C_1 consisting of 2π-periodic functions into $M \cap C_2$, then there exist an ϵ_4, $0 < \epsilon_4 \leqq \epsilon_1$, and a function $u(t,x,\gamma_0,\epsilon)$, continuous in t,x,ϵ and having continuous second derivatives with respect to t,x for $|\epsilon| \leqq \epsilon_4$, $0 \leqq t$, $x \leqq 2\pi$, such that $u(t,x,\gamma_0,0) = \gamma_0$ and $u(t,x,\gamma_0,\epsilon)$ satisfies (2.8) for $|\epsilon| \leqq \epsilon_4$. If g satisfies (A_3), then the same conclusions are valid provided there is a γ_0 in $M \cap C_2$, $\|\gamma_0\|_2 < a$ such that $H(\gamma_0,0) = 0$ and $H'(\gamma_0,0)$ has a continuous inverse which maps the subspace of C_2 consisting of 2π-periodic functions into $M \cap C_2$.

Remark. As we shall see in the applications, Theorem 4 in its present form is sometimes not convenient because of the condition on the inverse mapping being required to take all periodic functions of period 2π which are C_1(or C_2) into $M \cap C_2$. Actually, the mean values of the periodic functions in the domain of the inverse are not important as the following argument shows. The implicit function theorem could just as well be applied directly to the equations (2.15) which written out explicitly in terms of the function $H(\gamma,\epsilon)$ defined in (2.16) are

$$\tilde{H}(\gamma,\epsilon)(t,x) \overset{\text{def}}{=} H(\gamma,\epsilon)(x+t) - H(\gamma,\epsilon)(-x+t) = 0, \quad 0 \leqq x,t \leqq 2\pi.$$

Therefore, if we find a γ_0 in $M \cap C_2$ such that $\tilde{H}(\gamma_0,0) = 0$

and $\tilde{H}'(\gamma,0)$ has an inverse which maps $M \cap C_1$(or $M \cap C_2$) into $M \cap C_2$, then the implicit function theorem will imply the existence of a solution for ϵ small. Clearly,

$$(\tilde{H}'(\gamma,\epsilon)\Delta)(t,x) = (H'(\gamma,\epsilon)\Delta)(x+t)-(H'(\gamma,\epsilon)\Delta)(-x+t),$$

and finding the inverse of $\tilde{H}'(\gamma_0,0)$ is equivalent to solving the equation

$$(H'(\gamma_0,0)\Delta)(x+t)-(H'(\gamma_0,0)\Delta)(-x+t) = q(x+t)-q(-x+t)$$

$$0 \leqq x,\ t \leqq 2\pi$$

where q is an arbitrary 2π-periodic function in C_1(or C_2) whose mean value obviously is of no importance.

Another convenient form for the bifurcation equations is the following

Theorem 5. If g satisfies (A_1), then a necessary and sufficient condition (in the sense described above) for the existence of a solution of (2.8) is the existence of an ϵ_4, $0 < \epsilon_4 \leqq \epsilon_1$ and a function $\gamma(\epsilon)$ in $M \cap C_2$, $\|\gamma(\epsilon)\|_2 \leqq a$, $0 \leqq |\epsilon| \leqq \epsilon_4$, such that

(2.19) $$\int_0^{2\pi}\int_0^{\pi} G(t,x,\Gamma(\gamma(\epsilon),\epsilon))\Delta(t,x)dtdx = 0 \text{ for all } \Delta \text{ in } M \cap C_2,$$

where $\Gamma(\gamma, \epsilon)$ is given in Theorem 1 and $G(t,x,\varphi)$ in (2.12).

The proof of this follow immediately from Lemma 1 and the fact that γ must satisfy (2.16) in order to have a solution of (2.8). This type of criterion for the existence of a solution has been used by Rabinowitz [5], but directly on (2.8). Knowing that one need only solve (2.19) for the specific $\Gamma(\gamma, \epsilon)$ given in Theorem 1 should lead to some simplification in the proof of Rabinowitz.

2.3. Examples.

2.3.1. Consider the equation

$$u_{tt}-u_{xx} = \epsilon\,[u_t + bu + cu^3 + f(t,x)] \tag{2.20}$$

where b, c are constants, $b \neq 0$ and f satisfies (A_2) and (2.9). We will apply Theorem 4 to obtain for $|\epsilon|$ and $|c|$ small the existence of a solution $u(t,x,\epsilon)$ of (2.20) which is 2π-periodic in t, x and odd in x. The functions $H(\gamma,0)$, $H'(\gamma,0)\Delta$ in (2.17), (2.18) are easily seen to be

$$H(\gamma,0)(y) = p'(y)+[b+cm(p^2)]p(y)+cp^3(y)-cm(p^3)+h(y), \tag{2.21}$$

$$(H'(\gamma,0)\Delta)(y) = q'(y)+[b+6cm(p^2)]q(y)-3c[m(p^2q)-2m(pq)], \tag{2.22}$$

$$0 \leqq y \leqq 2\pi$$

where $\gamma(t,x) = p(x+t)-p(-x+t), \Delta(t,x) = q(x+t)-q(-x+t)$, $p(y), q(y)$ are 2π-periodic in y, $p'(y) = dp(y)/dy$, $q'(y) = dq(y)/dy$, $m(\varphi) = \int_0^{2\pi}\varphi(y)dy/2\pi$, and $m(p) = m(q) = 0$, and $h(y) = (1/2\pi)\int_0^{2\pi}f(s,y-s)ds$.

We need to show that equation (2.21) has a solution γ_0 in $M \cap C_2$ and that for this γ_0, the operator $H'(\gamma_0,0)$ defined by (2.22) has a continuous inverse which maps the subspace of C_1 consisting of 2π-periodic functions into $M \cap C_2$. It seems to be difficult to solve this problem in general, se we take a particular case; namely, c small. For arbitrary constants k, ℓ, $k > 0$, consider the equation

$$(2.23) \qquad p'(y) + (b+ck)p(y)+cp^3(y) - c\ell + h(y) = 0$$

For c sufficiently small, equation (2.23) has a unique 2π-periodic solution $p(y,k,\ell,c)$ satisfying

$$p(y,k,\ell,c) = -\int_0^\infty e^{-bu}h(y-u)du + O(c) \quad \text{as} \quad c \to 0,$$

where we have taken $b > 0$ for definiteness. If $b < 0$, then the same remark holds except a different integral is used. If this function p is to yield a solution of $H(\gamma,0) = 0$, then k, ℓ must satisfy the equations $k = m(p^2(\cdot,k,\ell,c))$, $\ell = m(p^3(\cdot,k,\ell,c))$, or

$$(2.24) \qquad \begin{aligned} &k - \frac{1}{2\pi}\int_0^{2\pi}\left[\int_0^\infty e^{-bu}h(y-u)du\right]^2 dy + O(c) = 0 \\ &\ell + \frac{1}{2\pi}\int_0^{2\pi}\left[\int_0^\infty e^{-bu}h(y-u)du\right]^3 dy + O(c) = 0 \end{aligned}$$

where the symbols $O(c)$ designate terms which approach zero as $c \to 0$. But equations (2.24) obviously have a solution for c sufficiently small and, thus, $H(\gamma,0) = 0$ has a solution γ_0 in $M \cap C_2$ for small c. The same type of argument shows that $H'(\gamma_0,0)$ has a continuous inverse of the desired type for c small. Theorem 4 them implies the existence of a solution of (2.20) which is 2π-periodic in t,x and odd in x.

2.3.2. Consider the equation

$$u_{tt}-u_{xx} = \epsilon[-u_t^3 + f(t,x)] \tag{2.25}$$

where f satisfies (A_2) and (2.9). We shall show by an application of Theorem 4 that for ϵ sufficiently small, equation (2.25) has a unique solution $u(t,x,\epsilon)$ which is 2π-periodic in t,x and odd in x provided that

$$h(y) \overset{\mathrm{def}}{\equiv} \frac{1}{2\pi}\int_0^{2\pi} f(s,y-s)ds, \quad h(y) \not\equiv 0,\ 0 \leqq y \leqq 2\pi ,$$

is an odd function of y. If $\gamma(t,x) = p(x+t)-p(-x+t), p(y+2\pi) = p(y)$ for all y, and if φ is any 2π-periodic function define

$$\alpha(y) = dp(y)/dy\ ,\ m(\varphi) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(s)ds.$$

A few simple computations yield from (2.17)

$$H(\gamma,0) = \frac{1}{2\pi}\int_0^{2\pi}\{[-\alpha^3(y)-3\alpha^2(y)\alpha(-y+2s)+3\alpha(y)\alpha^2(-y+2s)-\alpha^3(-y+2s)] + f(s,y-s)\}ds$$

$$= -\alpha^3(y)-3\alpha(y)m(\alpha^2) + m(\alpha^3) + h(y) \quad , \quad 0 \leqq y \leqq 2\pi .$$

If $H(\gamma,0) = 0$, then $h(-y) = -h(y)$ implies that

$$(2.26) \qquad -\alpha^3(y)-\alpha^3(-y)-3m(\alpha^2)[\alpha(y)+\alpha(-y)]+2m(\alpha^3) = 0 \ , \ 0 \leqq y \leqq 2\pi$$

and an integration from 0 to 2π yields $m(\alpha^3) = 0$. Also, if $m(\alpha^3) = 0$, then (2.26) yields $\alpha(y) = -\alpha(-y)$, $0 \leqq y \leqq 2$. Consequently, the equation $H(\gamma,0) = 0$ has a solution if and only if the equation

$$(2.27) \qquad \alpha^3(y) + 3m(\alpha^2)\alpha(y)-h(y) = 0 \ , \quad 0 \leqq y \leqq 2\pi,$$

has an odd solution.

For any constant $k > 0$, consider the equation

$$(2.28) \qquad \alpha^3(y) + 3k\alpha(y) - h(y) = 0 \ , \ 0 \leqq y \leqq 2\pi.$$

Since the discriminant of the polynomial on the left hand side of (2.28) is negative, there is a unique real solution $\alpha(y,k)$ of (2.28) given by

$$(2.29)\qquad \begin{aligned} 2^{1/3}\,\alpha(y,k) &= [h(y) + \beta(y,k)]^{1/3} + [h(y) - \beta(y,k)]^{1/3} \\ \beta(y,k) &\overset{\text{def}}{=} [h^2(y) + 4k^3]^{1/2}, \quad 0 \leqq y \leqq 2, \end{aligned}$$

and $\alpha(-y,k) = -\alpha(y,k)$. For this to be a solution of (2.27), k must be equal to $m[\alpha^2(\cdot,k)]$. Carrying out this computation, we obtain the result that k must satisfy

$$F(k) = 0$$

$$(2.30)\qquad F(k) \overset{\text{def}}{=} 3k - \frac{1}{2^{8/3}\pi}\int_0^{2\pi}\{[h(y)+\beta(y,k)]^{2/3} + [h(y)-\beta(y,k)]^{2/3}\}dy .$$

Another simple computation yields

$$\frac{dF(k)}{dk} = 3 + \frac{2^{2/3}k}{\pi}\int_0^{\pi}\frac{1}{\beta(y,k)}\;\{[h(y)+\beta(y,k)]^{1/3}-[h(y)-\beta(y,k)]^{1/3}\}dy.$$

For $k > 0$, this integrand is always positive and, thus, $dF/dk > 0$ for $k > 0$. Since $F(0) < 0$ and $dF(k)/dk \to \infty$ as $k \to \infty$, it follows that (2.30) has a unique solution k^*. The function $\alpha(y,k^*)$

given by (2.29) is therefore the unique real solution of (2.28). Since $\alpha(-y,k^*) = -\alpha(y,k^*)$, it follows that any primitive p of α yields a solution γ of $H(\gamma,0) = 0$ and it is unique. Before completing the proof of existence of a solution of (2.25), we make some remarks.

Notice that all constants and functions involved in this analysis could be obtained very easily on a computer. Also, once the constant k^* is found for which $F(k^*) = 0$, then one can also find a Fourier series solution of (2.28) with $k = k^*$. In general, this method of computation should be much easier than using double Fourier series in t and x in the original equation (2.25). However, it is questionable as to the merits of using any Fourier series at all, since knowing k^* yields by simple numerical integration a primitive p of $\alpha(y,k^*)$ form (2.29), and an approximate solution $\gamma(t,x) = p(x+t)-p(-x+t)$ of (2.25) to order ϵ as the superposition of two traveling waves. Another remark that seems to be interesting is that the constant $k^* = m[\alpha^2(\cdot,k^*)]$ is determined without knowing anything about the solution. Such constants should in general have some physical significance. For this particular case, one shows that up to terms of order ϵ, the total energy in the oscillatory motion at time t,

$$\int_0^{\pi} [\gamma_t^2(t,x) + \gamma_x^2(t,x)]dx \quad ,$$

is proportional to k^* and that the kinetic energy at $t = 0$,

$$\int_0^{\pi}[\gamma_t^2(0,x)]dx$$

is proportional to k^*. The integrations are taken from 0 to π only because of the original problem of the string with length π which was fastened at the ends 0 and π.

To complete the proof of the existence of a solution, we need to show that $H'(\gamma,0)$ given in (2.18) has a bounded inverse which maps the subspace of 2π-periodic functions in C_1 into $M \cap C_2$. We are going to make use of the remark following Theorem 4 where it was pointed out that the mean values of the functions in the domain of the inverse are not important. Letting $\Delta(t,x) = q(x+t) - q(-x+t)$, $\beta(s) = dq(s)/ds$, and $\gamma(t,x) = p(x+t)-p(-x+t)$, $\alpha(s) = dp(s)/ds$, where α satisfies (2.27), we obtain from (2.18) that

$$-\frac{1}{3}(H'(\gamma,0)\Delta)(y) = 2k\beta(y) + \frac{\alpha(y)}{\pi}\int_0^{2\pi}\alpha(s)\beta(s)ds -$$

$$-\frac{1}{2\pi}\int_0^{2\pi}\alpha^2(s)\beta(s)ds ,$$

where $k = m(\alpha^2)$. Our first problem is to solve the equation

$$\beta(y) + \frac{\alpha(y)}{2k\pi}\int_0^{2\pi}\alpha(s)\beta(s)ds - \frac{1}{4k\pi}\int_0^{2\pi}\alpha^2(s)\beta(s)ds = e(y), \tag{2.31}$$

$$0 \leq y \leq 2\pi ,$$

where $e(y)$ is a 2π-periodic function in C_1 which is arbitrary except we are allowed to choose the mean value in any manner whatsoever.

We choose our class of 2π-periodic functions e in the following manner. If $\tilde{e}$ is an arbitrary 2π-periodic function in C_1 with $m(\tilde{e}) = 0$, then

$$e = \tilde{e} - \frac{1}{4\pi k} \int_0^{2\pi} \alpha^2(s)\tilde{e}(s)ds$$

is an admissible 2π-periodic e. For any e of this form equation (2.31) has a unique solution given by

$$\beta(y) = e(y) - \frac{\alpha(y)}{4\pi k} \int_0^{2\pi} \alpha(s)e(s)ds + \frac{1}{2\pi k} \int_0^{2\pi} \alpha^2(s)e(s)ds \ .$$

Notice that $m(\beta) = 0$ and, therefore, $q(y) = \int^y \beta$ is 2π-periodic and yields a function Δ in $M \cap C_2$. Also, there exists a k such that $\|\Delta\| \leq k\|e\|_1$ for all e and we have proved our result.

3. The characteristic problem for the hyperbolic equation.

3.1. The linear characteristic problem. Consider the classes of functions S and N defined by

$$(3.1)\qquad \begin{aligned} S &= \{\varphi : \varphi(x + 2\pi, y) = \varphi(x,y) = \varphi(x, y + 2\pi)\}, \\ N &= \{\varphi : \varphi(x,y) = \alpha(x) + \beta(y)\} . \end{aligned}$$

The decomposition of elements in N is not unique, but it can be made so by arbitrarily specifying that either α or β has average zero over a period. For any φ in $S \cap C_k$, $k \geqq 0$, we define the element $P\varphi$ in $N \cap C_k$, $k \geqq 0$, by

$$(3.2)\qquad (P\varphi)(x,y) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(x,\eta)d\eta + \frac{1}{2\pi}\int_0^{2\pi} \varphi(\xi,y)d\xi - \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \varphi(\xi,\eta)d\xi d\eta.$$

It is clear that P is a projection operator of $S \cap C_k$ into $N \cap C_k$. Finally, if the set $N^\perp$ is defined by

$$(3.3)\qquad N^\perp = \{\varphi \text{ in } S \cap C_o : P\varphi = 0\} ,$$

then $N^\perp \cap (N \cap C_k) = \{0\}$ for all k.

The operator $P\varphi$ is the same as the one used by Cesari [2] except in his notation, he denoted this by $(P\varphi)(x,y) = m(x) + n(y) - \mu$.

<u>Lemma 4</u>. The following statements are equivalent:

i) $\varphi \in N^{\perp}$;

ii) $\varphi \in S \cap C_o$, $\int_0^{2\pi}\varphi(x,\eta)d\eta = 0$, $\int_0^{2\pi}\varphi(\xi,y)d\xi = 0$,

$0 \leqq x,\ y \leqq 2\pi$;

iii) $\int_0^{2\pi}\int_0^{2\pi}\psi(x,y)\varphi(x,y)dxdy = 0$ for all ψ in $N \cap C_o$.

<u>Proof</u>: The fact that i) is equivalent to ii) follows immediately from the definition. One shows that ii) and iii) are equivalent by using integration by parts to obtain

$$\int_0^{2\pi}\int_0^{2\pi} [\alpha(x)+\beta(y)]\varphi(x,y)dxdy = \int_0^{2\pi} \alpha(x)\ [\ \int_0^{2\pi} \varphi(x,\eta)d\eta]dx$$

$$= \int_0^{2\pi} \beta(y)[\int_0^{2\pi} \varphi(\xi,y)d\xi]dy$$

for all ψ in $N \cap C_o$, φ in $S \cap C_o$, $\psi(x,y) = \alpha(x)+\beta(y)$. The result then follows immediately.

The significance of the above definitions lies in the following lemmas.

<u>Lemma 5</u>. The set $N \cap C_1$ coincides with the solutions of the boundary value problem

$$(3.4)\qquad \begin{aligned} &u_{xy} = 0 \\ &u(x + 2\pi,y) = u(x,y) = u(x,y + 2\pi) \\ &-\infty < x,\ y < \infty \end{aligned}$$

Lemma 6. For a given integer k and a given φ in $S \cap C_k$, the boundary value problem

$$(3.5)\qquad \begin{aligned} &u_{xy} = \varphi(x,y) \\ &u(x+2\pi, y) = u(x,y) = u(x, y+2\pi) \\ &-\infty < x\ ,\ y < \infty \end{aligned}$$

has a solution if and only if $\varphi \in N^{\perp} \cap C_k$. Furthermore, if $\varphi \in N^{\perp} \cap C_k$, then there exists a unique solution of (3.5) which belongs to $N^{\perp} \cap C_{k+1}$. If this unique solution is designated by $\mathcal{L}(x,y)\varphi$, $0 \leqq x,y \leqq 2\pi$, then $\mathcal{L}(\cdot,\cdot)$ is a linear operator mapping $N^{\perp} \cap C_k$ into $N^{\perp} \cap C_{k+1}$ and there is a constant L such that

$$(3.6)\qquad \|\mathcal{L}(\cdot,\cdot)\varphi\|_{k+1} \leqq L\,\|\varphi\|_k\ .$$

Proof: The proof of the necessity in the first part of the lemma follows simply by integrating the differential equation and using the periodicity of u_x, u_y. For the sufficiency, put $u(x,y) = \int_0^x \int_0^y \varphi(\xi,\eta)d\xi d\eta$ and verify the periodicity directly. The uniqueness of a solution of (3.5) in $N^{\perp}$ follows because the difference of two such solutions would be in $N \cap C_1$ and $N^{\perp} \cap (N \cap C_1) = \{0\}$. Define

$$(3.17)\qquad \begin{aligned} \mathcal{L}(x,y)\varphi \overset{\text{def}}{=} &\int_0^x\int_0^y \varphi(\xi,\eta)d\xi d\eta - \frac{1}{2\pi}\int_0^{2\pi}\Big[\int_0^s\int_0^y \varphi(\xi,\eta)d\xi d\eta\Big]ds \\ &- \frac{1}{2\pi}\int_0^{2\pi}\Big[\int_0^x\int_0^r \varphi(\xi,\eta)d\xi d\eta\Big]dr + \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\Big[\int_0^s\int_0^r \varphi(\xi,\eta)d\xi d\eta\Big]drds \end{aligned}$$

It is clear that $\mathcal{L}(\cdot,\cdot)\varphi \in N^{\perp} \cap C_{k+1}$ and that there exist a constant L such that (3.6) is satisfied. This completes the proof of the lemma.

3.2. The nonlinear characteristic problem. Consider the problem

$$(3.8)\qquad \begin{aligned} &u_{xy} = \epsilon\, f(x,y,u,u_x,u_y) \\ &u(x+2\pi,y) = u(x,y) = u(x,y+2\pi) \end{aligned}$$

where ϵ is a real parameter and $f(x,y,u,p,q)$ is periodic in x and y of period 2π and continuous in x,y,u,p,q and locally lipschitizian in u,p,q in a region $\Omega(R)$ given by

$$(3.9)\qquad \Omega(R) = \{x,y,u,p,q : \ 0 \leqq x,y \leqq 2\pi \ , \ |u|+|p|+|q| < R\} \ .$$

For a fixed $\gamma \in N \cap C_1$ and for given positive constants a,b, $a < b < R$, let

$$(3.10)\qquad S_1(\gamma,a,b) = \{\psi \in S \cap C_1 : P\psi = \gamma,\ \|\gamma\|_1 \leqq a,\ \|\psi\|_1 \leqq b\} \ .$$

If φ is a given function in $S_1(\gamma,a,b)$, then we will let $F(\cdot,\cdot,\varphi)$ designate the function

$$(3.11)\qquad F(x,y,\varphi) \overset{\text{def}}{=} f(x,y,\varphi(x,y),\varphi_x(x,y),\varphi_y(x,y)),\ 0 \leqq x,y \leqq 2\pi.$$

<u>Theorem 6</u>. For any given positive constants $a < b < R$, there is an $\epsilon_1 > 0$ with the following property: corresponding to each $\gamma \in N \cap C_1$, $\|\gamma\|_1 \leq a$ and to each ϵ, $|\epsilon| \leq \epsilon_1$, there is a unique function $\Gamma = \Gamma(\gamma, \epsilon)$ in $S_1(\gamma, a, b)$ such that $\Gamma_{xy}(x,y)$ is continuous in x, y and satisfies

$$(3.12) \qquad \Gamma_{xy} = \epsilon\, F(x,y,\Gamma) - \epsilon\, PF(\cdot,\cdot,\Gamma)(x,y)$$

where F is defined in (3.11). The function $\Gamma(\gamma,\epsilon)$ can be obtained by the method of successive approximations

$$(3.13) \qquad \begin{aligned} &u^{(0)} = \gamma \\ &u^{(n+1)}(x,y) = \gamma(x,y) + \epsilon\, \mathcal{L}(x,y)(I-P)F(\cdot,\cdot,u^{(n)}), \\ &n = 0,1,2,\ldots \end{aligned}$$

where $\mathcal{L}(\cdot,\cdot)$ is defined in Lemma 6. Finally $\Gamma(\gamma,\epsilon)$ is continuous in γ, ϵ and lipschitzian in γ uniformly with respect to ϵ for $\|\gamma\|_1 \leq a$, $|\epsilon| \leq \epsilon_1$, $\Gamma(\gamma, 0) = \gamma$.

<u>Theorem 7</u>. Let $a < b < R$, ϵ_1, $\Gamma(\gamma,\epsilon)$ be the quantities given in Theorem 6. If there exist an $\epsilon_2 \leq \epsilon_1$ and a function $\gamma(\epsilon)$ in $N \cap C_1$, $|\gamma(\epsilon)| \leq a$, $|\epsilon| \leq \epsilon_1$, such that

$$(3.14) \qquad P\,F(\cdot,\cdot,\Gamma(\gamma(\epsilon),\epsilon)) = 0$$

then $\Gamma(\gamma(\epsilon),\epsilon)$ is a solution of (3.8) for $|\epsilon| \leqq \epsilon_2$. Conversely, if (3.8) has a solution $u(x,y,\epsilon)$ which is continuous in x,y,ϵ together with u_x, u_y, u_{xy} for $0 \leqq x,y \leqq 2\pi$, $|\epsilon| \leqq \epsilon_2$, $\|u(\cdot,\cdot,\epsilon)\|_1 < b$, $\|Pu(\cdot,\cdot,\epsilon)\|_1 \leqq a$ for $0 \leqq |\epsilon| \leqq \epsilon_2$, then $u(x,y,\epsilon) = \Gamma(x,y,\gamma(\epsilon),\epsilon)$ where Γ is the function given in Theorem 6, $Pu(\cdot,\cdot,\epsilon) = \gamma(\epsilon)$ and $\gamma(\epsilon)$ satisfies (3.14).

<u>Theorem 8</u>. Let $a < b < R$, ϵ_1, $\Gamma(\gamma,\epsilon)$ be as in Theorem 6. If $f(x,y,u,p,q)$ is Lipschitz continuously differentiable with respect to u,p,q in $\Omega(R)$, then there exists an ϵ_3, $0 < \epsilon_3 \leqq \epsilon_1$ such that $\Gamma(\gamma,\epsilon)$ is continuously differentiable with respect to γ for $\|\gamma\|_1 < a$, $|\epsilon| \leqq \epsilon_3$. Furthermore, the derivative of $\Gamma(\gamma,\epsilon)$ with respect to γ at $\epsilon = 0$ is the identity operator.

The proofs of Theorems 6,7,8 are exactly the same as the proofs of Theorem 1,2,3 if one replaces $M, Q, g, T_2(\gamma,a,b), G, \mathcal{K}$ in the preceeding proofs by $N, P, f, S_1(\gamma,a,b), F, \mathcal{L}$, respectively. Of course, the estimates are made in C_1 with the aid of Lemma 6.

Equations (3.14) are called the bifurcation equations or determining equations for problem (3.8) and a solution $\gamma(\epsilon)$ of these equations which belongs to $N \cap C_1$ is a necessary and sufficient condition (in the sense described by Theorem 2) for the existence of a solution to (3.8) for ϵ sufficiently small.

From Lemma 4, equations (3.14) are equivalent to the following:

$$H(\gamma, \epsilon) = 0 \ ,$$

$$(3.15) \qquad H(\gamma,\epsilon)(x,y) \stackrel{\text{def}}{=} \int_0^{2\pi} F(x,\eta,\Gamma(\gamma,\epsilon))d\eta + \int_0^{2\pi} F(\xi,y,\Gamma(\gamma,\epsilon))d\xi \ ,$$

$$0 \leqq x,y \leqq 2\pi \ ,$$

where F is defined in (3.11). Since $\Gamma(\gamma,0) = \gamma$, the "first approximation" to these equations are

$$H(\gamma,0) = 0 \ ,$$

$$(3.16) \qquad H(\gamma,0)(x,y) = \int_0^{2\pi} F(x,\eta,\gamma)d\eta + \int_0^{2\pi} F(\xi,y,\gamma)d\xi \ ,$$

$$0 \leqq x,y \leqq 2\pi \ .$$

If the conditions of Theorem 3 are satisfied, then the function $H(\gamma,\epsilon)$ defined in (3.15) is differentiable with respect to γ. If we designate the derivative with respect to γ by $H'(\gamma,\epsilon)$, then it is easy to show that

$$[H'(\gamma,\epsilon)\Delta](x,y) = \int_0^{2\pi} \text{grad } f(x,\eta,\overline{\gamma}(x,\eta))\cdot\overline{\Delta}\,(x,\eta)d\eta$$

$$+ \int_0^{2\pi} \text{grad } f(\xi,y,\overline{\gamma}(\xi,y))\cdot\overline{\Delta}\,(\xi,y)d\xi \ ,$$

$$(3.17)$$

$$0 \leqq x,y \leqq 2\pi$$

$$\text{grad } f \stackrel{\text{def}}{=} (f_u, f_p, f_q), \ \overline{\gamma} \stackrel{\text{def}}{=} (\gamma, \gamma_x, \gamma_y), \ \overline{\Delta} \stackrel{\text{def}}{=} (\Delta, \Delta_x, \Delta_y),$$

where f_u, f_p, f_q denote the partial deriviatives of $f(x,y,u,p,q)$ with respect to u,p,q, respectively.

If $f(x,y,u,p,q)$ depends explictly upon p,q then $H(\gamma,\epsilon)$, $H'(\gamma,\epsilon)$ in (3.16),(3.17) are continuous mappings of $N \cap C_1$ into $N \cap C_0$. If f depends only upon x,y,u, then $H(\gamma,\epsilon), H'(\gamma,\epsilon)$ are continuous mappings of $N \cap C_1$ into $N \cap C_1$.

By using these remarks, the implicit function theorem in Banach spaces (see [4]) and Theorem 7, one easily deduces the following result.

Theorem 9. Suppose the conditions of Theorem 8 are satisfied and $H(\gamma,\epsilon)$, $H'(\gamma,\epsilon)$ are defined by (3.15),(3.17), respectively. If there is a γ_0 in $N \cap C_1$, $\|\gamma_0\|_1 < a$, such that $H(\gamma_0,0) = 0$ and the linear operator $H'(\gamma_0,0)$ has a continuous inverse taking $N \cap C_0$ into $N \cap C_1$, then there exist an $\epsilon_4 > 0$ and a function $u(x,y,\gamma_0,\epsilon)$ continuous in x,y,ϵ for $|\epsilon| \leqq \epsilon_4$, $0 \leqq x,y \leqq 2\pi$, $u(x,y,\gamma_0,0) = \gamma_0(x,y)$, such that $u(x,y,\gamma_0,\epsilon)$ satisfies problem (3.8). The same conclusion holds if f depends only upon x,y,u and there is a γ_0 in $N \cap C_1$, $\|\gamma_0\|_1 < a$, such that $H(\gamma_0,0) = 0$ and $H'(\gamma_0,0)$ has a continuous inverse mapping $N \cap C_1$ into $N \cap C_1$.

Notice that nothing would be changed in the above theory if the function f in (3.8) depended continuously upon a parameter ϵ. We will actually use the theory for this case in the example below.

3.3. Examples.

3.3.1. Consider the system (3.8) with

$$f(x,y,u,\epsilon) = \epsilon[\psi(x,y) + Cu + \epsilon\, g(x,y,u)] \tag{3.18}$$

where $C \neq 0$ is a constant, ψ, g are continuously differentiable with respect to x,y,u. For γ,Δ in $N \cap C_1$, $\gamma(x,y) = \alpha(x)+\beta(y)$, $\Delta(x,y) = a(x) + b(x)$, and the particular f in (3.18), we obtain from (3.16),(3.17) that

$$H(\gamma,0)(x,y) = 2\pi C[\alpha(x)+\beta(y)] + C[\int_0^{2\pi} \alpha(\xi)d\xi + \int_0^{2\pi} \beta(\eta)d\eta]$$

$$+ [\int_0^{2\pi} \psi(x,\eta)d\eta + \int_0^{2\pi} \psi(\xi,y)dy], \quad 0 \leq x,y \leq 2\pi,$$

$$[H'(\gamma,0)\Delta](x,y) = 2\pi C[a(x) + b(y)], \quad 0 \leq x,y \leq 2\pi.$$

Since $C \neq 0$, $H(\gamma,0) = 0$ if $2\pi C\, \alpha(x) = -\int_0^{2\pi} \psi(x,\eta)d\eta$, $2\pi C\, \beta(y) = -\int_0^{2\pi}\psi(\xi,y)d\xi - C\int_0^{2\pi}\alpha(\xi)d\xi$. The operator $H'(\gamma,0)$ obvioulsy has a continuous inverse mapping $N \cap C_1$ into $N \cap C_1$ and Theorem 4 implies the existence for ϵ small of a solution of (3.8) with f given in (3.18).

3.2.2. Consider the system (3.8) with

$$f(x,y,u,p,q,\epsilon) = \psi(x,y) + Cu + \psi_1(y)p + \psi_2(x)q + \epsilon g(x,y,u,p,q) \tag{3.19}$$

where ψ, ψ_1, ψ_2 are continuous and g satisfies the conditions of Theorem 8. We shall also suppose that $C \neq 0$ and the quantities

$$r = \int_0^{2\pi} \psi_1(\eta) d\eta \quad , \quad s = \int_0^{2\pi} \psi_2(\xi) d\xi$$

are different from zero.

For any γ, Δ in $N \cap C_1$, $\gamma(x,y) = \alpha(x)+\beta(y), \Delta(x,y) = a(x)+b(y)$, ($'$) denoting differentiation, $H(\gamma,0)(x,y) = E(x,\gamma)+G(y,\gamma)$, $H'(\gamma,0) = E'(\gamma) + G(\gamma)$, and f given in (3.19) we obtain from (3.16),(3.17) that

$$E(x,\gamma) = 2\pi C\alpha(x) + r\alpha'(x) + \int_0^{2\pi} \psi(x,\eta)d\eta + C\int_0^{2\pi} \beta(\eta)d\eta,$$

$$G(y,\gamma) = 2\pi C\beta(y) + s\beta'(y) + \int_0^{2\pi} \psi(\xi,y)dy + C\int_0^{2\pi} \alpha(\xi)d\xi,$$

$$E'(\gamma) = 2\pi Ca(x) + ra'(x),$$

$$G'(\gamma) = 2\pi Cb(y) + sb'(y),$$

for $0 \leqq x,y \leqq 2\pi$. From the remark after the definition in (3.1) it is no loss in generality to assume that $\int_0^{2\pi}\beta(\eta)d\eta = 0$. The relation $H(\gamma,0) = 0$ is equivalent to $E(x,\gamma) = 0$, $G(y,\gamma) = 0$ and the above conditions on ψ, ψ_1, ψ_2 and C imply that the relation $E(x,\gamma) = 0$ has a unique 2π-periodic function $\alpha(x)$ which has a continuous first derivative. Having determined α, the relation $G(y,\gamma) = 0$ determines a unique 2π-periodic function $\beta(y)$ which has continuous first derivative. This same argument shows that $H'(\gamma,0)$ has a continuous inverse mapping $N \cap C_0$ into $N \cap C_1$ and Theorem 9 then implies the existence for ϵ small of a solution of problem (3.8) with f given in (3.19).

Remark. Cesari [2] discusses (3.8) with f satisfying (3.19) only under the assertion $C \neq 0$ and asserts the same conclusion as in section 3.3.2. Notice that we need $r, s \neq 0$ if ψ_1, ψ_2 are not identically zero. If ψ_1, ψ_2 are identically zero, we need g depending only on x, y, and u as in (3.18).

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